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1985 J. Phys. A: Math. Gen. 18 2639

(http://iopscience.iop.org/0305-4470/18/14/015)

J. Phys. A: Math. Gen. 18 (1985) 2639-2646. Printed in Great Britain

Symmetries of the pre-Klein–Gordon bundle: a Lagrangian analysis of quantum relativistic symmetry

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Received 18 February 1985

Abstract. We discuss here the Lagrangian formulation of a group theoretical quantisation procedure where the rest mass of a scalar relativistic particle is introduced by putting a restriction (the mass shell condition) on the manifold of a group, G_{15}, defining a larger symmetry. This group is obtained through one of the possible contractions of the conformal group and, because of its non-trivial cohomology, G_{15} allows for a central *quantum* U(1)-extension. Besides illustrating the appearance of the mass term in the relativistic Lagrangian as a consequence of reducing the symmetry, we obtain the Klein-Gordon 'probability' current as the one associated to a relativistic central symmetry.

1. Introduction and results

It is well known that the problem of quantising† a relativistic particle in a geometric way presents some specific difficulties which are absent for its Galilean counterpart. These difficulties ultimately have their root in the relative nature of time in special relativity. Indeed, it is a special feature of the different 10-parameter kinematical groups [1, 2] that they fall into two different classes, of relative and absolute time. Those of relative time, like the Poincaré group $\mathscr{P}$, have a trivial second cohomology group, while those of absolute time, like the Galilei group G, have a non-trivial cohomology $H_0^2(\mathrm{G}, \mathrm{U}(1)) = \mathbb{R}^+$ [1]. Thus, the Galilei group can be centrally extended by U(1) [3-5] giving as a result an 11-parameter 'quantum group' $\tilde{\mathrm{G}}_{(m)}$, which incorporates a commutator of the form

$$[\text{translation, boosts}] = m \times (\text{central U(1) generator}). \qquad (1.1)$$

In its quantum version, (1.1) reads $[p_i, x^j] = -\mathrm{i}\hbar\delta^j_i I$ and allows for a complete and well defined geometric quantisation process on the group manifold of $\tilde{\mathrm{G}}_{(m)}$ [6]. In contrast, no such extension exists for $\mathscr{P}$ [7], (1.1) is not included in its Lie algebra (and hence the difficulties of a relativistic position operator [8, 9]), and the geometric quantisation of $\mathscr{P}$ cannot be directly performed.

Associated with the above cohomological difference between the Poincaré and the Galilei groups are the peculiarities of the 'non-relativistic' limit. While the well known contraction process [10]—in the form of the $c \to \infty$ limit—brings $\mathscr{P}$ to G, this limit cannot be directly performed on the Klein-Gordon equation to obtain the Schrödinger

† By quantisation we obviously mean here first quantisation, i.e. the derivation of the associated relativistic wave equation and the expression of the basic quantum operators acting on the wavefunctions.

0305-4470/85/142639+08$02.25

equation without previously subtracting the rest energy, which otherwise would go to infinity. Finally, the apparent contradiction which presents the existence of a necessarily trivial (direct product) extension of $\mathscr{P}$ by U(1) which contracts to a central extension $\tilde{G}_{(m)}$ of G is only solved after realising that the 'non-relativistic' limit may generate cohomology [11, 12].

The above features of the relativistic geometry show that, to perform a geometric quantisation of the relativistic case in a way similar to the Galilean one [6], one has to resort to a group larger than $\mathscr{P}$† which does not preserve $p^\mu p_\mu = (mc)^2$; the mass shell condition corresponds to an orbit of $\mathscr{P}$ on this larger group. This program has been recently performed [13] by taking as the starting point the 15-parameter group G_{15} which is derived by contraction from the conformal group. As we shall discuss in § 2, this group can be extended by U(1) and accordingly leads to a generalisation of (1.1) in terms of a 4-component off-shell 'position' operator.

In this paper we shall restrict ourselves to analyse some aspects of the quantisation process from the Lagrangian point of view. In this approach the relativistic wavefunction ψ is a solution of an equation invariant under a *ray* [3] representation of the above group G_{15}, a group which acts on an absolute time τ besides acting on the ordinary Minkowski coordinates x^μ. It will be shown that the representation of G_{15} is projective because the true invariance group is a U(1)-central extension of this group, $\tilde{G}_{16}$, which exists because of the non-trivial cohomology of G_{15}. The Lagrangian density leading to the aforementioned equation, which originally does not contain a mass term, acquires a term of the form $\psi^*\psi$ when the mass shell condition is imposed; this also reduces the symmetry group from G_{15} down to $\mathscr{P}$.

This paper is organised as follows. In § 2, and in a vector-bundle framework, we describe the Lagrangian formalism invariant under a central extension of G_{15}, and derive the expression of the sixteen conserved Noether charges associated with the $\tilde{G}_{16}$ symmetry after giving the group law of that extension and its action on the 'pre-Klein-Gordon' bundle. Section 3 is devoted to describing the dynamics on the group manifold of $\tilde{G}_{16}$, including some brief comments on a group quantisation formalism [6, 13]. After discussing how the partial breaking of the symmetry implied by the restriction to an orbit in $\tilde{G}_{16}$ introduces the mass shell condition, we finally consider in § 4 the introduction of this constraint into the Lagrangian formalism and the attainment of the familiar Klein-Gordon (KG) symmetries. In this way, a mass term appears in the Lagrangian density and the KG probability current is directly obtained, and the former absolute time parameter becomes the proper time of the free particle.

2. Off-shell Klein–Gordon Lagrangian formalism

Our task here is to find an invariant Lagrangian under the projective representations of a group G_{15} (or under the representations of its quantum extension by U(1), $\tilde{G}_{16}$) whose structure is compatible with the quantum interpretation of the Poisson brackets of its associated conserved magnitudes (see [14] for the case of the Galilei group) and which, of course, leads to an equation of motion generalising the KG one. The only requirement which is needed to have this group acting on Minkowski space $\mathscr{M}$ is to enlarge $\mathscr{M}$ ($\approx \mathbb{R}^4$) to $\mathbb{R}^5$ by means of an additional coordinate τ which, when the

† We shall not consider here graded Lie groups; see in this respect [19] and references therein.

reconstruction to the physical situation is made, will become the proper time of the free particle.

An essential feature of the group G_{15} [13] is thus its non-trivial cohomological structure; the wave equation of the system associated with G_{15} is invariant only if one allows for projective representations. This situation, which is also encountered in the Galilei case, has to be considered as inherent to these quantum systems and not as an anomaly. In fact, the phases which appear in the transformation of the wavefunctions are incorporated into a new family of transformations by extending the group by U(1) by means of a non-trivial system of exponents [3] ξ characterising the extended or *quantum* group $\tilde{G}$, which is the true underlying symmetry group of the quantum theory. In this way, the group law of $\tilde{G}_{16}$ is obtained by adding the composition law of the U(1) part, parametrised by θ (or by $\zeta \equiv e^{i\theta}$),

$$\begin{aligned} &(g'', \theta'') \equiv (g', \theta') * (g, \theta) = (g' * g, \theta' + \theta + \xi(g', g)), \\ &\tilde{g} \equiv (g, \theta) \in \tilde{G}_{16}, \end{aligned} \tag{2.1}$$

to the group law $g'' = g' * g$ of G_{15}.

Characterising the elements of G_{15} by $g = (b, A^{\mu}, u^{\nu}, \Lambda)$ where $b \in \mathbb{R}$, $A^{\mu} \in \mathbb{R}^4$, $u^{\mu} \in \mathbb{R}^4$, $\Lambda \in SO(3, 1)$, its action on Minkowski space $\mathcal{M}$ extended by τ is given by

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}{}_{.\nu} x^{\nu} + u^{\mu}\tau + A^{\mu} \\ \tau' &= \tau + b. \end{aligned} \tag{2.2}$$

The multiplicative action of $\zeta \equiv e^{i\theta} \in U(1)$ on the complex numbers allows us now to obtain a projective representation of G_{15} or a linear representation of $\tilde{G}_{16}$ on the cross sections ψ of the trivial vector bundle $E \equiv \mathbb{R}^5 \times \mathbb{C} \xrightarrow{\pi} \mathbb{R}^5 = \mathcal{M} \times \mathbb{R}$ (the *pre-Klein-Gordon bundle*) by means of the expression

$$\begin{aligned} &[U(b, A^{\mu}, u^{\nu}, \Lambda, \theta)\psi](x', \tau') \\ &\qquad = \exp[-iM(\tfrac{1}{2}u^{\mu}u_{\mu}\tau + u_{\mu}\Lambda^{\mu}{}_{.\nu}x^{\nu} - \theta/M)\psi(x, \tau)], \end{aligned} \tag{2.3}$$

where M is a parameter with dimensions of mass ($[u] = LT^{-1}, [\tau] = T$) which characterises the central extension $\tilde{G}_{16}$. Two consecutive transformations reveal the group law of $\tilde{G}_{16}$:

$$\begin{aligned} b'' &= b' + b \\ A''^{\mu} &= A'^{\mu} + \Lambda'^{\mu}{}_{.\nu} A^{\nu} + u'^{\mu} b \\ \Lambda'' &= \Lambda' \Lambda \\ u''^{\mu} &= u'^{\mu} + \Lambda'^{\mu}{}_{.\nu} u^{\nu} \\ \theta'' &= \theta' + \theta + \xi(g', g) \end{aligned} \tag{2.4}$$

where the non-trivial local exponent $\xi(g', g)$ or 2-cocycle is given by

$$\xi(g', g) = -M(\tfrac{1}{2}u'_{\mu}u'^{\mu}b + u'_{\mu}\Lambda'^{\mu}{}_{.\nu}A^{\nu}). \tag{2.5}$$

Although (2.5) is uniquely determined by (2.3), different exponents differing in a trivial one $\xi_{cob} = \delta(g' * g) - \delta(g') - \delta(g)$ (i.e. in a coboundary; δ is a real function on G) define the same extension and equivalent projective representations.

The infinitesimal generators of the action of $\tilde{G}_{16}$ on the bundle $E \xrightarrow{\pi} \mathcal{M} \times \mathbb{R}$ parametrised by the coordinates $(\tau, x^{\mu}, \psi, \psi^*)$ where ψ, ψ^* are the coordinates of $\mathbb{C}$, is obtained easily from (2.2) and (2.3) with the result

$$\begin{aligned}
&\tilde{X}_{(b)} = \partial/\partial\tau \\
&\tilde{X}_{(A^\mu)} = \partial/\partial x^\mu \\
&\tilde{X}_{(J^{\mu\nu})} = \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon\, \partial/\partial x^\sigma \\
&\tilde{X}_{(u^\mu)} = \tau\, \partial/\partial x^\mu - M x_\mu (\mathrm{i}\psi\, \partial/\partial\psi - \mathrm{i}\psi^*\, \partial/\partial\psi^*) \\
&\tilde{X}_{(\theta)} = \mathrm{i}\psi\, \partial/\partial\psi - \mathrm{i}\psi^*\, \partial/\partial\psi^*.
\end{aligned} \tag{2.6}$$

Among these generators there are 4+4 vector fields $\tilde{X}_{(u^\mu)}$, $\tilde{X}_{(A^\nu)}$ whose commutator is $g_{\mu\nu}$ times the central generator and which possess the adequate Lorentz transformation properties as to give, through their associated Noether conserved charges, 'position' coordinates and conjugate momenta. The full set of commutators is the following

$$\begin{aligned}
&[\tilde{X}_{(b)}, \tilde{X}_{(A^\mu)}] = 0 && [\tilde{X}_{(A^\mu)}, \tilde{X}_{(A^\nu)}] = 0 \\
&[\tilde{X}_{(b)}, \tilde{X}_{(J^{\mu\nu})}] = 0 && [\tilde{X}_{(A^\mu)}, \tilde{X}_{(J^{\nu\rho})}] = \delta^{\varepsilon\sigma}_{\nu\rho} g_{\mu\varepsilon} \tilde{X}_{(A^\sigma)} \\
&[\tilde{X}_{(b)}, \tilde{X}_{(u^\nu)}] = \tilde{X}_{(A^\nu)} && [\tilde{X}_{(A^\mu)}, \tilde{X}_{(u^\nu)}] = -g_{\mu\nu} M \tilde{X}_{(\theta)} \\
&[\tilde{X}_{(J^{\mu\nu})}, \tilde{X}_{(u^\rho)}] = -\delta^{\varepsilon\sigma}_{\mu\nu} g_{\rho\varepsilon} \tilde{X}_{(u^\sigma)} && [\tilde{X}_{(u^\mu)}, \tilde{X}_{(u^\nu)}] = 0 \\
&[\tilde{X}_{(\theta)}, \text{any } \tilde{X}] = 0.
\end{aligned} \tag{2.7}$$

It should be noted that by *putting* $M = 0$ above one recovers the group law of $\mathrm{G}_{15} \otimes \mathrm{U}(1)$. It is however (2.7) which permits an isomorphism among the Lie and the Poisson brackets; the fact that the $[\tilde{X}_{(A^\mu)}, \tilde{X}_{(u^\nu)}]$ cannot be made zero in $\tilde{\mathrm{G}}_{16}$ is a consequence of its being a true central extension of G_{15}.

We wish to find now a Lagrangian density invariant under the action of $\tilde{\mathrm{G}}_{16}$. We consider Lagrangian densities depending on the fields and their first derivatives. In the above vector bundle framework, fields are given by cross sections $\psi \in \Gamma(E)$; fields and derivatives of fields are incorporated by taking cross sections of the bundle $J^1(E)$ of the 1-jets of E, on which the condition of being 1-jet cross sections is imposed†; Lagrangians are given as functions $\mathscr{L}$: $J^1(E) \to \mathbb{R}$, $\mathscr{L}(\tau, x^\mu, \psi, \psi_\mu, \psi_\tau, \psi^*, \psi^*_\nu, \psi^*_\tau)$. To decide whether such a Lagrangian is invariant under $\tilde{\mathrm{G}}_{16}$ it is necessary to lift the action (2.6) on E to an action on $J^1(E)$. This is uniquely done [15] by imposing on the 'prolonged' vector fields $\tilde{X}^1$ the conditions of being projectable onto the vector fields $\tilde{X}$ on E of (2.6) and of preserving the 1-jet prolongation cross sections, i.e. of preserving the structure 1-forms

$$\begin{aligned}
&\theta = \mathrm{d}\psi - \psi_\mu\, \mathrm{d}x^\mu - \psi_\tau\, \mathrm{d}\tau \\
&\theta^* = \mathrm{d}\psi^* - \psi^*_\mu\, \mathrm{d}x^\mu - \psi^*_\tau\, \mathrm{d}\tau
\end{aligned} \tag{2.8}$$

which define when a cross section $\psi^1 = (\psi, \psi_\mu, \psi_\tau, \psi^*, \psi^*_\mu, \psi^*_\tau) \in \Gamma(J^1(E))$ is a 1-jet prolongation of $\psi \in \Gamma(E)$ by means of the conditions $\theta|_{\psi^1} = 0$, $\theta^*|_{\psi^1} = 0$ (which clearly imply $\psi_\mu = \partial_\mu\psi$, $\psi_\tau = \partial_\tau\psi$ etc). Thus, by imposing to the general vector field

$$\tilde{X}^1 = \tilde{X} + [X_{\psi_\mu}\, \partial/\partial\psi_\mu + X_{\psi_\tau}\, \partial/\partial\psi_\tau + \mathrm{cc}] \tag{2.9}$$

the conditions

$$L_{\tilde{X}^1}\theta = \theta, \qquad L_{\tilde{X}^1}\theta^* = \theta^* \tag{2.10}$$

† A description of the variational principles on jet-bundles is given in [15, 16]. We use the same notation here.

where L means Lie derivative, we may determine [15] the unknown X_{ψ_μ}, X_{ψ_τ} components in (2.9) in terms of those of $\tilde{X}$. The resulting 1-jet prolongated vector fields $\bar{\tilde{X}}^1$ are given by

$$\begin{aligned}
&\bar{\tilde{X}}^1_{(b)} = \tilde{X}_{(b)} \qquad \bar{\tilde{X}}^1_{(A^\mu)} = \tilde{X}_{(A^\mu)} \\
&\bar{\tilde{X}}^1_{(J^{\mu\nu})} = \tilde{X}_{(J^{\mu\nu})} + \delta^{\varepsilon\sigma}_{\mu\nu}\psi_\varepsilon\, \partial/\partial\psi^\sigma + \delta^{\varepsilon\sigma}_{\mu\nu}\psi^*_\varepsilon\, \partial/\partial\psi^{\sigma *} \\
&\bar{\tilde{X}}^1_{(u^\mu)} = \tilde{X}_{(u^\mu)} - (\psi_\mu + \mathrm{i}Mx_\mu\psi_\tau)\, \partial/\partial\psi_\tau - \mathrm{i}M(g_{\mu\nu}\psi + x_\mu\psi_\nu)\, \partial/\partial\psi_\nu \\
&\qquad\qquad - (\psi_\mu - \mathrm{i}Mx_\mu\psi^*_\tau)\, \partial/\partial\psi^*_\tau + \mathrm{i}M(g_{\mu\nu}\psi^* + x_\mu\psi^*_\nu)\, \partial/\partial\psi^*_\nu \\
&\bar{\tilde{X}}^1_{(\theta)} = \tilde{X}_\theta + \mathrm{i}(\psi_\tau\, \partial/\partial\psi_\tau - \psi^*_\tau\, \partial/\partial\psi^*_\tau) + \mathrm{i}(\psi_\mu\, \partial/\partial\psi_\mu - \psi^*_\mu\, \partial/\partial\psi^*_\mu).
\end{aligned} \tag{2.11}$$

Since $\tilde{\mathrm{G}}_{16}$ has trivial cohomology, we may now look for a strictly invariant Lagrangian. The simplest solution for

$$L_{\bar{\tilde{X}}^1}\mathscr{L} = 0 \qquad \forall \bar{\tilde{X}}^1 \text{ in (2.11)} \tag{2.12}$$

is provided by

$$\mathscr{L} = \tfrac{1}{2}M^{-1}\psi^*_\mu\psi^\mu + \tfrac{1}{2}\mathrm{i}(\psi^*\psi_\tau - \psi^*_\tau\psi). \tag{2.13}$$

The ordinary Hamilton principle on $J^1(E)$ gives for (2.13) the Euler-Lagrange equation

$$\mathrm{i}\,\partial_\tau\psi - \tfrac{1}{2}M^{-1}\,\partial_\mu\,\partial^\mu\psi = 0 \tag{2.14}$$

similar to a Schrödinger equation in five dimensions.

The expression for the conserved currents associated with the different transformations of $\tilde{\mathrm{G}}_{16}$ is obtained from that of the general Noether current. Putting $x^A = (\tau, x^\mu)$, $\mu = 0, 1, 2, 3$, this is given by [15]

$$j^A_{(\alpha)} = (X^\psi_{(\alpha)} - \psi_B X^B_{(\alpha)})\, \partial\mathscr{L}/\partial\psi_A + \mathrm{HC} + \mathscr{L}X^A_{(\alpha)}, \tag{2.15}$$

$\alpha = b, A^\mu, J^{\mu\nu}, u^\mu, \theta$; see (2.6). (Note that the 1-jet prolongations $\bar{\tilde{X}}^1$ are necessary only to check the invariance of $\mathscr{L}$ and that their additional components X_{ψ_μ}, X_{ψ_τ} do not appear in the Noether currents.) Because $\partial_A j^A = 0$, the τ component of the currents is the density which gives rise to the τ conserved charges. These densities are

$$\begin{aligned}
&j^\tau_{(b)} = \tfrac{1}{2}M^{-1}\psi^*_\mu\psi^\mu \equiv \mathscr{H} \\
&j^\tau_{(A^\mu)} = \tfrac{1}{2}\mathrm{i}\psi^*\psi_\mu + \mathrm{HC} \equiv \mathscr{P}_\mu \\
&j^\tau_{(J^{\mu\nu})} = \tfrac{1}{2}\mathrm{i}\psi^*\delta^{\varepsilon\sigma}_{\mu\nu}x_\varepsilon\psi_\sigma + \mathrm{HC} \equiv \mathscr{J}_{\mu\nu} \\
&j^\tau_{(u^\mu)} = \tfrac{1}{2}M\psi^*x_\mu\psi - \tfrac{1}{2}\mathrm{i}\psi^*\tau\psi_\mu + \mathrm{HC} \equiv \mathscr{K}_\mu.
\end{aligned} \tag{2.16}$$

Finally, the full current associated to the phase transformation is

$$j^A_{(\theta)} = (-\psi^*\psi, -\tfrac{1}{2}\mathrm{i}M^{-1}(\psi^*\psi^\mu - \psi^{*\mu}\psi)) \equiv -(\rho, j^\mu). \tag{2.17}$$

Although we shall have to wait for the addition of the mass/shell condition to interpret physically the charge densities of (2.16), it is worth noticing here that, formally, $\mathscr{H}$ plays the role of the covariant Hamiltonian, $\mathscr{P}_\mu$ that of four-momentum density ($\mathscr{P}_0$ being associated with the customary Hamiltonian), $\mathscr{J}_{\mu\nu}$ that of a generalised angular momentum and $\mathscr{K}_\mu$ that of the true position ('centre-of-mass') density.

3. Group manifold dynamics and the mass shell condition

Before we impose a constraint on the Lagrangian theory of § 2 aiming to put the solutions of (2.14) on the mass shell $p^\mu p_\mu = M^2c^2$, it is necessary to analyse the meaning of this condition in a group framework since it involves more physical variables than just those appearing in the Lagrangian (2.13). Moreover, we wish that *all* the operations carried out on the above Lagrangian formalism be directly associated with the invariance group in the same sense that, for example, one would not admit other basic (i.e. canonical) observables than those associated with the Noether invariants of the $\tilde{G}_{16}$ symmetries.

From the $\tilde{G}_{16}$ group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$ (equations (2.4) and (2.5)) two sets of invariant vector fields, left and right, may be derived by computing $\partial\tilde{g}''/\partial\tilde{g}|_{\tilde{g}=e}$ and $\partial\tilde{g}''/\partial\tilde{g}'|_{\tilde{g}'=e}$ respectively. We explicitly give here the right-invariant ones

$$\begin{aligned}
&\tilde{X}^{\mathrm{R}}_{(b)} = \partial/\partial b \qquad \tilde{X}^{\mathrm{R}}_{(A^\mu)} = \partial/\partial A^\mu \\
&\tilde{X}^{\mathrm{R}}_{(J^{0i})} = \tfrac{1}{2}(1+\alpha^2)^{-1/2}[(1+2\alpha^2)\delta^j_i - \alpha_i\alpha^j]\,\partial/\partial\alpha^j \\
&\qquad + \tfrac{1}{2}(1+\alpha^2)^{-1/2}[(1-\varepsilon^2)^{1/2}\eta^j_{\cdot ik} - \eta^j_{\cdot sn}\eta^n_{\cdot ki}\varepsilon^s]\alpha^k\,\partial/\partial\varepsilon^j \\
&\qquad + A^0\,\partial/\partial A^i + A^i\,\partial/\partial A^0 + u^0\,\partial/\partial u^i + u^i\,\partial/\partial u^0 \qquad (3.1)\\
&\tilde{X}^{\mathrm{R}}_{(J^{jk}\equiv J^l)} = \tfrac{1}{2}[(1-\varepsilon^2)^{1/2}\delta^m_i + \eta^m_{\cdot ni}\varepsilon^n]\,\partial/\partial\varepsilon^n + \eta^m_{\cdot ni}(\alpha^n\,\partial/\partial\alpha^m + A^n\,\partial/\partial A^m + u^n\,\partial/\partial u^m) \\
&\tilde{X}^{\mathrm{R}}_{(u^\mu)} = \partial/\partial u^\mu + b\,\partial/\partial A^\mu + MA_\mu\Xi \qquad \tilde{X}^{\mathrm{R}}_{(\rho)} = \mathrm{i}\zeta\,\partial/\partial\zeta \equiv \Xi.
\end{aligned}$$

In (3.1) the Lorentz group is parametrised by $(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, where $\boldsymbol{\varepsilon}$ corresponds to a rotation of angle $\varphi = 2\sin^{-1}|\boldsymbol{\varepsilon}|$ around the axis $\boldsymbol{\varepsilon}$, and $\boldsymbol{\alpha}$ characterises a boost such that, with $\chi = \tanh^{-1}|v/c|$, $\chi = 2\sinh^{-1}|\boldsymbol{\alpha}|$.

The generators of (3.1) were utilised [13] in the group manifold quantisation formalism [6, 13] to define quantum operators acting on wavefunctions, which in turn were defined from U(1)-equivariant (i.e. $\Xi\cdot\psi = \mathrm{i}\psi$, $\Xi\cdot\psi^* = -\mathrm{i}\psi^*$) functions on $\tilde{G}_{16}$. Accordingly, these operators should agree with the physical generators (2.6) which act on the pre-Klein-Gordon bundle $(\tau, x^\mu, \psi, \psi^*)$. Indeed, eliminating $(u^\mu, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, changing M by $-M$† and identifying $(b, A^\mu, \zeta, \zeta^*)$ with $(\tau, x^\mu, \psi, \psi^*)$ respectively we obtain (2.6) from (3.1). This process of removing some variables is in practice rather involved although a standard one in geometric quantisation schemes, where it corresponds to defining a *polarisation* [17]. In our specific situation, it consists in polarising the U(1)-function on $\tilde{G}_{16}$ by the set of conditions

$$\tilde{X}^{\mathrm{L}}_{(b)}\cdot\psi = 0, \qquad \tilde{X}^{\mathrm{L}}_{(A^\mu)}\cdot\psi = 0, \qquad \tilde{X}^{\mathrm{L}}_{\boldsymbol{\varepsilon}}\cdot\psi = 0, \qquad \tilde{X}^{\mathrm{L}}_{\boldsymbol{\alpha}}\cdot\psi = 0 \tag{3.2}$$

(associated with the *polarisation subalgebra* [6] $\langle\tilde{X}^{\mathrm{L}}_{(b)}, \tilde{X}^{\mathrm{L}}_{(A^\mu)}, \tilde{X}^{\mathrm{L}}_{(\boldsymbol{\varepsilon})}, \tilde{X}^{\mathrm{L}}_{(\boldsymbol{\alpha})}\rangle$) which imply that $\psi = \psi(b, A^\mu, u^\nu, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$ is actually of the form

$$\psi = \exp(-\mathrm{i}MA^\mu u_\mu)\exp(\tfrac{1}{2}\mathrm{i}Mu^\nu u_\nu b)\varphi(u^0, \boldsymbol{u}) \tag{3.3}$$

then a Fourier transformation of the 'momentum' space wavefunction $\varphi(u^0, \boldsymbol{u})$ gives

† The expression of the quantum operators associated with group transformations are derived by comparing $\psi(x)$ with the infinitesimally transformed $\psi'(x)$ (this variation is given by the Lie derivative L_X). Thus, to compute the vector fields (2.6) we have to consider the transformation law in the form $\psi'(x,\tau) = \exp\mathrm{i}\xi(g, g^{-1}(x,\tau))\cdot\psi(g^{-1}(x,\tau))$ rather than in the form (2.3), i.e. $\psi'(g(x,\tau)) = \exp\mathrm{i}\xi(g;(x,\tau))\cdot\psi(x,\tau)$. This leads to the above rule (note that the ξ on the bundle E, (2.3), is simply obtained from that on the group, (2.5), by substituting x, τ for A^μ, b in the unprimed argument g and by adding the term in θ).

the 'stationary' wavefunction associated with (2.14) on which the generators of (2.6) other than $\tilde{X}^{\mathrm{L}}_{(b)}$ act. This vector field,

$$\tilde{X}^{\mathrm{L}}_{(b)} = \partial/\partial b + u^\mu\, \partial/\partial A^\mu + \tfrac{1}{2} M u^\nu u_\nu \Xi, \tag{3.4}$$

generates in fact the Euler-Lagrange equation (2.14) after identifying b with τ, A^μ with x^μ and $Mu_\mu\psi$ (which involves the 'momentum' Mu_μ) with $\mathrm{i}\psi_\mu$. This last identification is suggested by the Noether charge density $j^\tau_{(A^\mu)}$ (equation (2.16)) and the action of the quantum operator [13] $\hat{P}_\mu = \mathrm{i}\tilde{X}^{\mathrm{R}}_{(A^\mu)}$ on (3.3).

Finally, we now put the theory on the mass shell. On our group formalism this means putting the following restriction of the *unprimed* factor of the group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$

$$\left.\begin{aligned} u^0 &= c(1+2\alpha^2) \equiv p^0/M \\ \boldsymbol{u} &= 2c(1+\alpha^2)^{1/2}\boldsymbol{\alpha} \equiv \boldsymbol{p}/M \end{aligned}\right\} \Rightarrow u^\mu u_\mu = c^2 \tag{3.5}$$

where p^μ is the momentum acquired by a particle of rest energy Mc^2 boosted by $\boldsymbol{\alpha}$. This restriction, since it must transform the group law into a group action $m'' = g'm(m'', m \in \tilde{\mathrm{G}}_{16}|_{u^\mu = p^\mu/M})$, makes sense only for the elements g' which preserve the orbit $\tilde{\mathrm{G}}_{16}|_{u^\mu u_\mu = c^2}$ and, accordingly, for the right (action) vector fields [13] which operate on the wavefunctions now restricted to

$$\psi = \exp(-\mathrm{i}A^\mu p_\mu)\exp(\tfrac{1}{2}\mathrm{i}Mc^2 b)\varphi(p^0, \boldsymbol{p}). \tag{3.6}$$

The left vector field $\tilde{X}^{\mathrm{L}}_{(b)}$ (3.4) which gives the equation of motion is also well behaved in the restriction process and leads to the familiar KG equation. It also now says how b is related to the other group parameters: b turns out to be the proper time of the particle. This is obtained by deriving the classical trajectory by integration of the evolution vector field $X^{\mathrm{L}}_{(b)}$; indeed $\mathrm{d}A^\mu/\mathrm{d}b = u^\mu$ implies, after using (3.5), $\mathrm{d}A^\mu/\mathrm{d}b = p^\mu/M$.

4. From the extended Minkowski space to Minkowski space: Lagrangian dynamics on the mass shell

As discussed in the previous section, the mass shell condition on the manifold of $\tilde{\mathrm{G}}_{16}$ is a constraint affecting several group parameters $(u^\mu \sim p^\mu)$ which do not appear directly in the Lagrangian formalism. (It is true, nevertheless, that Mu^μ may be understood as the eigenvalue of the operator $\hat{P}_{(\mu)} = \mathrm{i}\tilde{X}^{\mathrm{R}}_{(A^\mu)}$ (equation (3.1)) acting on the manifold of solutions (3.6) or of (2.14) where the mass shell condition—which identifies Mu^μ with p^μ—has been imposed.)

It was also shown in § 3 that the vector field $\tilde{X}^{\mathrm{L}}_{(b)}$ was the generator of the evolution leading to the (Euler-Lagrange) wave equation. $\tilde{X}^{\mathrm{L}}_{(b)}$ admits the restriction to the mass shell and—by means of the identifications after (3.4)—its expression on the space $J^1(E)$ of definition of Lagrangian is given by

$$\partial/\partial\tau + p^\mu M^{-1}\,\partial/\partial x^\mu + \tfrac{1}{2}\mathrm{i}Mc^2\Xi. \tag{4.1}$$

Now, let us define the variational principle for cross sections $\Gamma(E)$ which are both 1-jet $(\psi_\mu = \partial_\mu\psi)$ and have null derivative under the action of (4.1). This means that in the Lagrangian density we may put

$$\psi_\tau = -p^\mu M^{-1}\psi_\mu - \tfrac{1}{2}\mathrm{i}Mc^2\psi. \tag{4.2}$$

This gives

$$\mathscr{L}_{\mathrm{KG}} \equiv -2M\mathscr{L}|_{\substack{\text{mass}\\ \text{shell}}} = \psi^*_\mu \psi^\mu - M^2c^2\psi^*\psi \tag{4.3}$$

which is the familiar KG Lagrangian density on $J^1(E)$ where now the *KG bundle E* is parametrised by (x^μ, ψ, ψ^*) [16].

In the same way, 12 out of the 16 current densities of (2.16), (2.17), pass to the restriction defining the (Poincaré)$\otimes$U(1) symmetry plus $j^\tau_{(\tau)}$. These clearly correspond to the 12 right-invariant vector fields of (2.11) which are well defined (act on the orbit $u^\mu u_\mu = c^2$) under the restriction. All these currents are associated with the standard Poincaré currents for the Poincaré group

$$\mathscr{P}^\mu = \psi^* p^\mu \psi \qquad \mathscr{J}^{\mu\nu} = \psi^*(x^\mu p^\nu - x^\nu p^\mu)\psi \tag{4.4}$$

plus $j^\tau_{(\tau)}$, whose charge defines the evolution associated with the 'invariant' Hamiltonian $\mathscr{H} \sim \psi^*_\mu \psi^\mu$, and the one associated with U(1) (equation (2.17)) which, because after imposing (4.2) we have $\partial_\tau(\psi^*\psi) = 0$, leads to the familiar Klein–Gordon continuity equation

$$\partial_\mu j^\mu = 0 \qquad j_\mu \equiv \tfrac{1}{2}\mathrm{i}M^{-1}(\psi^*\psi_\mu - \psi^*_\mu\psi). \tag{4.5}$$

We see thus how the ordinary conservation of the (non-positive definite) space integral of the Klein–Gordon 'probability' density j_0 is associated with the aforementioned U(1) factor. A similar analysis may be performed in the Galilean case [18] where the U(1) factor is the centre of the quantum group $\tilde{G}_{(m)}$.

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